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ON THE ANALYSIS OF THIN POROUS COATINGS*

E.V. KOVALENKO

A plane contact problem is considered for an elastic layer whose pores are filled with a viscous incompressible fluid. It is shown that in the case of a relatively small layer (coating) thickness its rheological properties can be modelled by equations of the Fuss-Winkler foundation with a bed operator coefficient (the analogue of the hereditary elasticity equations). The case of the impression of a parabolic stamp in a thin porous-elastic coating is investigated in detail. Asymptotic formulas are obtained for the fundamental contact interaction characteristics, namely, the settling of the foundation under the stamp, the contact domain, and the contact pressure, which hold for short and long times.

The experience of producing and using antifriction coatings in modern engineering results in the need to control their structure and functional properties. Among such coatings one should mention primarily porous-elastic coatings whose surface is antifrictional by virtue of its ability to absorb oil and then to release it under loading. Moreover, the theory of the deformation of porous-elastic bodies is convenient for describing a number of features of material production by porous metallurgy methods /1/. The principles of this theory were

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developed quite long ago /2/, but the solution of specific mixed problems has received insufficient attention. In the main these are problems of the theory of the consolidation of water-saturated media /1, 3, 4/, where a porous-elastic half-plane or half-space were selected as the foundation.

1. Within the framework of the plane theory of elasticity (plane strain) we examine the problem of the action of a normal load $q(x)H(t)$ ($H(t)$ is the Heaviside function) distributed over a section $|x| \leqslant a$ on the upper boundary of a porous elastic layer $(0 \leqslant y \leqslant h)$ clamped rigidly along the foundation. We will describe the rheological properties of the medium by equations of the Biot model /2/ by considering the motion of a viscous (η is the coefficient of viscosity) compressible fluid in the pores subject to the Darcy filtration law with permeability factor k

$$\mu \Delta \mathbf{u} + (\mu + \lambda_c) \operatorname{grad} e - \alpha M \operatorname{grad} \zeta = 0 \tag{1.1}$$

$$\frac{\partial \zeta}{\partial t} = \frac{kM_c}{\eta} \Delta \zeta \quad \left(M_c = \frac{M(2\mu + \lambda)}{2\mu + \lambda_c}\right) \tag{1.2}$$

$$p = -\alpha M e + M\zeta, \quad e = \operatorname{div} \mathbf{u}, \quad \zeta = f \operatorname{div} (\mathbf{u} - \mathbf{U}) \tag{1.3}$$

$$\tau_{ij} = 2\mu e_{ij} + \delta_{ij} (\lambda_c e - \alpha M \zeta), \quad \lambda_c = \lambda + \alpha^2 M \tag{1.4}$$

Here $\mathbf{u} = \{u, v\}$, $\mathbf{U}$ are the displacement vectors of points of the elastic skeleton and the fluid, respectively, p is the hydrostatic fluid pressure in the pores, f is the porosity, τ_{ij} is the stress tensor in the porous medium, e_{ij} is the strain tensor in the elastic skeleton, and μ, λ, α and M are the elastic coefficients of the porous medium whose physical meaning and methods of determination are considered in /5/.

We assume the layer surface $y = h$ to be completely permeable and its foundation to be absolutely impermeable. Then the boundary conditions of the problem formulated can be written in the form

$$\begin{gathered} y = h\colon\ p = 0,\ \tau_{12} = 0 \ (|x| < \infty) \\ \tau_{22} = -q(x) H(t) \ (|x| \leqslant a),\ \tau_{22} = 0 \ (|x| > a) \\ y = 0\colon\ u = v = \partial p / \partial y = 0 \ (|x| < \infty) \end{gathered} \tag{1.5}$$

where the stresses in the layer equal zero at infinity while the initial condition corresponding to no instantaneous bulk strains in the skeleton has the form

$$e(x, y, 0) = 0 \tag{1.6}$$

To solve problem (1.1)-(1.6), just as was done in /6/, we introduce two unknown functions $E(x, y, t)$ and $S(x, y)$ associated with the displacement vector components in the elastic skeleton u, v and the pressure p by the expressions

$$\begin{gathered} u = E_{,x} + yS_{,x}, \quad v = E_{,y} + yS_{,y} - S \\ p\alpha = (2\mu + \lambda) \Delta E + 2\mu S_{,y}, \quad e = \Delta E \end{gathered} \tag{1.7}$$

Substituting (1.7) into (1.1), (1.2) and (1.4) we obtain

$$\Delta S = 0, \quad \Delta E_{,t} = c \Delta^2 E \quad (c = kM_c/\eta) \tag{1.8}$$

$$\begin{gathered} \tau_{11} = 2\mu(-E_{,xx} + yS_{,yy} - S_{,y}), \\ \tau_{12} = 2\mu(E_{,xy} + yS_{,xy}) \end{gathered} \tag{1.9}$$

Now using (1.7) and (1.9), we convert the boundary conditions (1.5) and the initial condition (1.6) into the form

$$\begin{gathered} y = h\colon\ (2\mu + \lambda) \Delta E + 2\mu S_{,y} = 0, \quad E_{,xy} + hS_{,xy} = 0 \\ -E_{,xx} + hS_{,yy} - S_{,y} = -{}^1/_2 q(x) \mu^{-1} [(H(x + a) + \\ H(x - a)] H(t) \end{gathered} \tag{1.10}$$

$$\begin{gathered} y = 0\colon\ E_{,x} = E_{,y} - S = (2\mu + \lambda) \Delta E_{,y} + 2\mu S_{,yy} = 0 \ (|x| < \infty) \\ t = 0\colon\ \Delta E = 0 \end{gathered} \tag{1.11}$$

The boundary-value problem (1.8)-(1.11) can be solved exactly by using Fourier integral transformations in the x coordinate and the Laplace-Carson transformation in time t /7/. Omitting the computations, we obtain

$$v^L(x,h,s) = -\frac{1}{\pi\mu}\int_{-a}^{a} q(\xi)\,d\xi \int_0^\infty K(u,s)\cos\frac{u}{h}(\xi - x)\,du \tag{1.12}$$

$$K(u,s) = D_1(u,s)/[uD_2(u,s)]$$

$$D_1(u,s) = 2u^2 \operatorname{sh} u - u\gamma \operatorname{sh}\gamma + {}^1/_2\varepsilon^{-1}ums \operatorname{ch}\gamma + u^3\gamma^{-1}\operatorname{sh}^2 u \operatorname{sh}\gamma - {}^1/_4\varepsilon^{-1}[\gamma^2 - \nu(1-\nu)^{-1}u^2]\operatorname{sh} 2u \operatorname{ch}\gamma$$

$$D_2(u,s) = 4u^2(\operatorname{ch} u - u \operatorname{sh} u - \operatorname{ch}^2 u \operatorname{ch}\gamma) + u\gamma^{-1}\operatorname{sh}\gamma \times (u^2+\gamma^2)(\operatorname{sh} 2u + 2u) - \varepsilon^{-1}ms \operatorname{ch}\gamma(u^2 + \operatorname{ch}^2 u) - 4\varepsilon(ms)^{-1}u^3 \operatorname{ch} u\,[2u(\operatorname{ch} u \operatorname{ch}\gamma - 1) - (u^2+\gamma^2)\gamma^{-1}\operatorname{sh} u \operatorname{sh}\gamma]$$

$$\gamma^2 = u^2 + ms,\quad m = h^2/c,\quad \varepsilon = {}^1/_2(1-2\nu)(1-\nu)^{-1}$$

for the Laplace-Carson transforms of the vertical displacement of points of the upper face of the layer (ν is Poisson's ratio of the skeleton material).

Solving the problem formulated for $t = 0$ we write

$$v(x,h,0) = -\frac{1-\nu}{\pi\mu}\int_{-a}^{a} q(\xi)\,k\left(\frac{\xi - x}{h}\right)d\xi \tag{1.13}$$

$$k(z) = \int_0^\infty \frac{(2\varkappa \operatorname{sh} 2u - 4u)\cos uz\,du}{(2\varkappa \operatorname{ch} 2u + 4u^2 + \varkappa^2 + 1)\,u},\quad \varkappa = 3 - 4\nu$$

We introduce the parameter $\Lambda = ha^{-1}$ into the considerations and we will assume $\Lambda \ll 1$. Furthermore, simplifying the relations (1.12) and (1.13) asymptotically, i.e., neglecting terms of the order of Λ^2 and higher, that characterise the deformation of the elastic skeleton in them, we find

$$v^L(x,h,s) = -\frac{\varepsilon h \operatorname{th}\sqrt{ms}}{\mu\sqrt{ms}}\,q(x),\quad v(x,h,0) = -\frac{\varepsilon h}{\mu}q(x)\quad (|x| \leqslant a) \tag{1.14}$$

We conclude from (1.14) that a relatively thin porous elastic layer behaves under compression in the same way as a Fuss-Winkler foundation with a bed operator coefficient whose form can be determined by applying an inverse Laplace-Carson transformation to both sides of the first equality in (1.14). Taking account of the second relationship in (1.14), we will have

$$v(x,h,t) = -\frac{\varepsilon h}{\mu}q(x)\left[1 + \frac{1}{m}\int_0^t \theta_2\left(0, \frac{t-\tau}{m}\right)d\tau\right] \tag{1.15}$$

$$(|x| \leqslant a,\ t \geqslant 0)$$

$$\theta_2(u,x) = 2\sum_{n=0}^{\infty}\exp\left[-\pi^2\left(n+\frac{1}{2}\right)^2 x\right]\cos\pi(2n+1)u$$

($\theta_2(u,x)$ is the theta function). The solution for an instantaneous distributed load $\tau_{22} = -q(x)\,\delta(t)$ applied to a section $|x| \leqslant a$ of the layer boundary $y = h$ is obtained by differentiating (1.15) with respect to t

$$v_{,t}(x,h,t) = -\frac{\varepsilon h}{\mu}q(x)\left[\delta(t) + \frac{1}{m}\theta_2\left(0, \frac{t}{m}\right)\right] \tag{1.16}$$

where

$$\theta_2(0,t) = \begin{cases} 2\exp(-\pi^2 t/4), & t \to \infty \\ (\pi t)^{-1/2}, & t \to 0 \end{cases} \tag{1.17}$$

2. Knowing the function $v_{,t}(x,h,t)$ describing the settling of points of the foundation under an instantaneous distributed load, we study the contact problem of the impression of a parabolic stamp $y = x^2(2R)^{-1}$ by a force $P(t)$ into the surface of a thin elastic porous layer (1.1)-(1.4) clamped rigidly along the lower face. We assume here that the force $P(t)$ varies with time in the same manner as the halfwidth of the contact domain $a(t)$, of the stamp with the foundation, increases with time. In this case a function $t = b(a)$ inverse to $a = a(t)$ exists and its uniqueness enables us to use the quantity $a(t)$ as a time parameter /8/.

Introducing the dimensionless variables $x^* = xR^{-1},\ t^* = tm^{-1}$ and using the notation

$$a^*(t^*) = a(t)R^{-1},\ \beta^*(t^*) = \beta[a(t)]R^{-1},\ q^*(x^*, t^*) = \varepsilon hq[x, a(t)](\mu R)^{-1},\ N_0(t^*) = \varepsilon hP[a(t)](\mu R^2)^{-1}$$

(we omit the asterisk here), we can write the integral equation of the problem formulated in the unknown contact pressure $q(x, t)$ under the stamp in the form

$$q(x,t) + \int_0^t q(x,\tau)k(t-\tau)\,d\tau = \beta(t) - \frac{x^2}{2} \tag{2.1}$$

$$(0 \leqslant x \leqslant a(t),\ 0 \leqslant t < \infty),\ k(t) = \theta_2(0, t)$$

The quasi-equilibrium condition

$$N_0(t) = 2\int_0^a q(x,t)\,dx \tag{2.2}$$

and the relationship

$$q(x, t) = 0 \quad (x \geqslant a(t)) \tag{2.3}$$

for finding the unknown domain of stamp contact with the porous foundation must be added to (2.1) to close the formulation of the problem.

Note that (2.3) enables us to rewrite the integral Eqs.(2.1) in the form of the following system:

$$q(x,t) + \int_{\psi(x)}^t q(x,\tau)k(t-\tau)\,d\tau = \beta(t) - \frac{x^2}{2}\ (\psi(x) \leqslant t < \infty) \tag{2.4}$$

$$\psi(x) = \begin{cases} 0 & (0 \leqslant x \leqslant a_0) \\ b(x) & (a_0 < x \leqslant a) \end{cases},\quad a_0 = a(0) \tag{2.5}$$

to solve which we use a well-known algorithm /8/.

Setting $x = a(t)$ in (2.4) and taking condition (2.3) into account, we determine the foundation settling function

$$\beta(t) = {}^1/_2 a^2(t) \tag{2.6}$$

We integrate both sides of (2.1) with respect to x between the limits 0 and $a(t)$. Using expressions (2.2) and (2.6) and changing the order of integration in the integral obtained, which is legitimate /9/ since the contact domain is described by a function which increases monotonically with time, we will have

$$a^3(t) = \frac{3}{2}\left\{N_0(t) + \int_0^t N_0(\tau)k(t-\tau)\,d\tau\right\} \tag{2.7}$$

Furthermore, limiting ourselves to the modification $N_0(t) = N_0 = \text{const}$, by taking account of (1.17) we find from (2.7)

$$a^3(t) = 3N_0\left[1 - \frac{1}{\pi^2}\sum_{n=0}^{\infty}\frac{\exp[-\pi^2(n+{}^1/_2)^2 t]}{(n+{}^1/_2)^2}\right] \tag{2.8}$$

$$a^3(t) = \frac{3}{2}N_0\begin{cases} \{1 + 8\pi^{-2}[1 - \exp(-\pi^2 t/4)]\} & (t \to \infty) \\ (1 + 2\sqrt{t/\pi}) & (t \to 0) \end{cases}$$

The solution of the system of integral Eqs.(2.4) and (2.5) is obtained in /8/ for sufficiently long times and has the form

$$q(x,t) = \begin{cases} I(0,t) + {}^1/_2[a^2(t) - x^2] & (0 \leqslant x \leqslant a_0) \\ I(b(x), t) & (a_0 < x \leqslant a(t)) \end{cases} \tag{2.9}$$

$$I(\alpha, t) = -\int_\alpha^t \exp[-(2 + {}^1/_4\pi^2)(t-\tau)][a^2(\tau) - x^2]\,d\tau$$

$$b(x) = -4\pi^{-2}\ln[1 + {}^1/_8\pi^2(1 - {}^2/_3 x^3 N_0^{-1})]$$

As $t \to 0$ the solution can be found by using the Laplace-Carson integral transform with respect to time. Omitting the computations we write it in the form of (2.9), where

$$I(\alpha, t) = \int_{\alpha}^{t} \left[e^{t-\tau} \operatorname{erfc}\left(\sqrt{t-\tau}\right) - \frac{1}{\sqrt{\pi (t-\tau)}} \right] \frac{a^2(\tau) - x^2}{2} \, d\tau \tag{2.10}$$

$$b(x) = {}^1/_4\pi \left({}^2/_3 x^3 N_0^{-1} - 1\right)^2$$

Therefore, (2.6) and (2.8)-(2.10) yield the solution of the problem of the impression of a parabolic stamp into a thin porous elastic coating as $t \to \infty$ and $t \to 0$.

Let us clarify the following question: do the asymptotic formulas constructed for short and long times merge? Values of the quantities $a(t)(3N_0)^{-1/3}$ x 10^3 are presented in the table according to the exact solution (first line), computed by means of (2.8) as $t \to 0$ (the second line) and $t \to \infty$ (the third line). It is seen that the asymptotic form for short times behaves well up to practically $t = 1$ (the error of this solution as compared with the exact solution for $t = 1$ does not exceed 3.2%). At the same time, the asymptotic solution as $t \to \infty$ can be utilized when $t \geqslant 0.7$ (the maximum error of the results is not greater than 3.6%).

$t=0$	0.1	0.2	0.5	0.7	1	2	∞
794	879	909	959	975	989	999	1000
794	879	910	965	991	1021	1091	∞
794	838	870	923	941	955	966	967

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